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Vortex Cutting by a Blade, Part 1: General Theory and a Simple Solution

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A simplified theory for long-wave motion of vortex filaments with variable core radius is given, for small values of the ratio $\varepsilon = \max(\sigma/L)$ of core radius σ to axial length scale L, which offers a number of advantages for numerical computations of the vortex response. An analytical solution of this theory is then obtained for the axial flow within a straight vortex filament following cutting of the vortex by a flat blade at an angle of attack α. The mathematical structure of the solution is similar to the classic problem of impulsive piston motion in a shock tube, and analogies of shocks and expansion waves propagating on the vortex away from the blade are predicted. A simple expression for the vortex force on the blade is derived, which is found to be mainly dependent on the ratio of the vortex core radii on opposing sides of the blade.

Nomenclature

$\boldsymbol{a}_3$	= tangent vector to vortex axis
a_{33}	= square of the stretch of the vortex axis, $\boldsymbol{a}_3 \cdot \boldsymbol{a}_3$
b	= rate of translation of observational frame
C	= vortex axis
c	= vortex "sound" speed
F	= rate of axial momentum supply to a vortex discontinuity
F_B	= total magnitude of force on blade by vortex
F_p	= magnitude of force on blade due to pressure differences external to the vortex
F_s	= magnitude of force on blade due to suction of boundary-layer fluid into vortex
$\boldsymbol{f}$	= vortex "body force" per unit mass
J^+, J^-	= Riemann invariants for axial flow in vortex core
L	= vortex axial length scale
ℓ	= one-half radial force per unit mass on core surface
M	= rate of mass supply to a vortex discontinuity
$\boldsymbol{n}$	= vortex "contact force"
p	= integrated pressure over the core cross section
p_s	= average pressure about core circumference
p_∞	= pressure at infinity
$\boldsymbol{q}$	= relative normal velocity to vortex axis
$\boldsymbol{r}$	= position vector to vortex axis
s	= arclength along the vortex axis
t	= time
U	= speed of forward motion of the blade
$\boldsymbol{u}$	= total vortex external velocity (averaged about core circumference)
$\boldsymbol{u}_E$	= external prescribed velocity of vortex with components (u_E, v_E, w_E)
$\boldsymbol{u}_I$	= self-induced velocity of vortex with components (u_I, v_I, w_I)
$\boldsymbol{v}$	= vortex velocity with components (u, v, w)
W	= propagation speed of vortex shock
w_A	= relative axial velocity in vortex core
w_s	= speed of blade boundary-layer fluid sucked into vortex core
$\boldsymbol{X}$	= added mass force on the vortex
$\boldsymbol{Y}$	= buoyancy force on the vortex
x, y, z	= Cartesian coordinates
z_F, z_R	= front and rear heights of vortex expansion wave
z_0	= height of vortex-blade intersection
α	= angle of attack
Γ	= vortex circulation
δ_R	= Rosenhead cutoff constant
ε	= ratio of vortex core radius to axial length scale
κ	= curvature of the vortex axis
λ	= initial value of the vortex density
λ_i	= unit vectors in the tangent ($i = 3$), principal normal ($i = 1$), and binormal ($i = 2$) directions to the vortex axis
μ	= Rosenhead cutoff parameter
ν	= fluid kinematic viscosity
ξ	= Lagrangian coordinate on the vortex axis
ρ	= vortex density
ρ^*	= density of the fluid making up the vortex
σ	= vortex core radius
τ	= torsion of the vortex axis

Subscripts

B	= value near the blade
0	= ambient value

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I. Introduction

IN this and subsequent papers, the results are reported of a study on the problem of the cutting of a vortex filament by a blade, where the direction of relative motion between the vortex and the blade is nearly normal to the vortex axis. The study particularly examines the effects of axial flow within the vortex core and vortex bending on variation of the core radius, as well as the subsequent force exerted on the blade. The current paper deals with theoretical aspects of the problem, whereas in later papers an experimental and computational investigation of normal vortex cutting by a blade is performed to examine the degree of vortex bending due to interaction with the blade and to provide experimental verification of the main results of the present paper.

The problem of normal vortex cutting by a blade has been identified as a major noise source for certain types of helicopters due to passage of tail rotor blades through trailing tip vortices shed from blades of the main rotor.[1] Normal vortex cutting can similarly occur due to ingestion of ambient turbulence or of vortices shed from an upstream body by the blades of a helicopter main rotor or a jet engine fan.

Experiments on noise generation from vortex cutting have been performed by Ahmadi,[2] Schlinker and Amiet,[3] and Cary.[4] In these experiments, a trailing tip vortex is shed from a stationary airfoil in a wind tunnel and cut by a downstream rotor. The characteristics of the sound field due to the blade-vortex interaction were measured, and in some cases pressure measurements on the blade surface and flow visualization with smoke injection were also reported. In the flow visualization results,[4] a variation in core radius on the vortex just after cutting is observed, followed by disruption of the vortex. However, the vortex response is somewhat obscured in these experiments by interaction of the vortex being cut with the trailing tip vortex of the cutting blade. Some additional preliminary experimental results were reported in a talk by Weigand and Gharib[5] for the problem of the cutting of a vortex ring by a thin plate at zero angle of attack. They also observed variation of core radius and propagation of axial waves on the vortex away from the plate, which causes a "sloshing" type of wave motion on the vortex ring.

Theoretical predictions for sound generation by normal vortex cutting, based on either a prescribed blade loading or vortex response, have been given by Amiet[6] and Howe.[7–9] A study of rotor noise due to turbulence ingestion was performed by Simonich et al.,[10] with the related acoustic results reported in Amiet et al.,[11] using rapid distortion theory to determine the distortion of the vorticity field. Although such an approach may model the intensification of vortex lines due to stretching from the mean flow, it cannot account for the nonlinear vortex response to interaction with, or cutting by, the blade.

The current study is concerned with understanding the vortex response to normal blade interaction and cutting. In Sec. II, after recalling some background material, the author formulates a simplified theory for long-wave motion of a vortex core with variable core area that admits both axial flow and discontinuities along the vortex filament. This simplified theory has a number of advantages for numerical computation when compared with more general theories.[12–15] Details of the application of this theory to the numerical computation of normal vortex-blade interaction are given in Part 2. In Sec. III, an analytical solution of the theory is given for the problem of normal cutting of a straight vortex by a flat plate. An expression for the vortex force on the blade is derived in Sec. IV. Conclusions of the paper are given in Sec. V, along with a rough assessment of the effect of normal vortex cutting on noise generation by the blade.

II. Long-Wave Theory of Vortex Filaments with Variable Core Radius

The self-induced velocity $\boldsymbol{u}_I$ of a vortex filament with circulation Γ, whose central axis occupies a space curve C, is given by the usual cutoff approach[16] as

$$\boldsymbol{u}_I(\xi, t) = -\frac{\Gamma}{4\pi}\int_C \frac{\boldsymbol{r}(\xi, t) - \boldsymbol{r}(\xi', t)}{[|\boldsymbol{r}(\xi, t) - \boldsymbol{r}(\xi', t)|^2 + \mu^2(\xi, t)]^{3/2}} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\xi}(\xi', t)\,\mathrm{d}\xi' \tag{1}$$

where $\boldsymbol{r}(\xi, t)$ is the position vector of a material point denoted by Lagrangian variable ξ at time t. The Rosenhead cutoff μ of the singularity at $\xi' = \xi$ is used, where $\mu(\xi, t) = 2\delta_R \sigma(\xi, t)$ is proportional to the core radius σ. The coefficient δ_R depends on the vorticity distribution across the core, and for a hollow vortex is given by $\delta_R = (1/2)\exp(-1/2)$.

In a general vortex motion, different sections of the vortex axis C will be stretched by different amounts, resulting in a variation of core radius along C. An axial gradient in core radius causes a gradient in pressure on the core lateral surface, which in turn results in an axial force acting on the core fluid. This force produces an axial motion within the core that is not predicted by the usual cutoff approach. For many problems, such as those involving vortex cutting or vortex breakdown phenomena, this axial motion within the vortex core is one of the primary features of interest in the problem solution.

Two different theories that account for variation of core radius along C are given in Refs. 14 and 15, both of which reduce to similar equations for sufficiently long, weakly nonlinear axial motion along the vortex. Before discussing dynamical theories of vortex motion, it is necessary to recall some kinematical aspects of the space curve C. In particular, if $\mathrm{d}\xi$ is chosen as the length of an infinitesimal segment of C at some initial time, the vector $\boldsymbol{a}_3 = \partial \boldsymbol{r}/\partial\xi$ is everywhere tangent to C and its magnitude $(a_{33})^{1/2} = (\boldsymbol{a}_3 \cdot \boldsymbol{a}_3)^{1/2}$ is simply the stretch of an infinitesimal segment of C about a particle ξ at time t. The length $\mathrm{d}s$ at time t of a segment of C with initial length $\mathrm{d}\xi$ is given by

$$\mathrm{d}s = (a_{33})^{1/2}\,\mathrm{d}\xi \tag{2}$$

The unit tangent vector λ_3 to C and the unit principal normal and binormal vectors λ_1 and λ_2, respectively, are given by

$$\lambda_3 = \boldsymbol{a}_3/(a_{33})^{1/2}, \qquad \lambda_1 = \frac{\partial\lambda_3/\partial\xi}{|\partial\lambda_3/\partial\xi|}, \qquad \lambda_2 = \lambda_3 \times \lambda_1 \tag{3}$$

From the Serret-Frenet equations, we also have that

$$\frac{\partial\lambda_3}{\partial s} = \kappa\lambda_1 \tag{4a}$$

$$\frac{\partial\lambda_2}{\partial s} = -\tau\lambda_1 \tag{4b}$$

$$\frac{\partial\lambda_1}{\partial s} = \tau\lambda_2 - \kappa\lambda_3 \tag{4c}$$

where Eqs. (4a) and (4b) serve as definitions of curvature κ and torsion τ and Eq. (4c) is an identity.

We now associate with every point on C a "vortex density" ρ, defined by $\rho = \pi\sigma^2\rho^*$ where ρ^* is the density of the fluid composing the vortex core. The assumption that mass is conserved within the core, for an inviscid fluid, then implies that

$$\rho(a_{33})^{1/2} = \lambda \tag{5}$$

where $\lambda(\xi)$ is independent of time and equal to the initial value of ρ.

Conservation of momentum within the vortex core can be expressed by

$$\lambda\dot{\boldsymbol{v}} = \lambda\boldsymbol{f} + \frac{\partial\boldsymbol{n}}{\partial\xi} \tag{6}$$

where $\boldsymbol{v}$ is the velocity of material points on C, $\boldsymbol{f}$ acts at any point along C, and $\boldsymbol{n}$ acts on the endpoints of any material segment of C. A superposed dot denotes the material derivative

$$\dot{a} \equiv \frac{\partial a}{\partial t} + (\boldsymbol{v}\cdot\lambda_3)\left(\frac{\partial a}{\partial s}\right) \tag{7}$$

The left-hand side of Eq. (6) represents the rate of change of momentum of fluid within the vortex core per unit length of C. The force $\boldsymbol{f}$ is due to forces acting on the lateral core boundary, due both to flow relative to the core and to variation of the core radius along C. Based on arguments given in Ref. 13, an expression for $\boldsymbol{f}$ can be assumed as

$$\rho\boldsymbol{f} = \rho^*\Gamma\boldsymbol{q} \times \lambda_3 - \rho^*\pi\sigma\ell\,\frac{\partial}{\partial s}(\sigma^2\lambda_3) + \boldsymbol{X} + \boldsymbol{Y} \tag{8}$$

where $\boldsymbol{q}$ is the projection in the normal plane to C of the difference between the fluid velocity $\boldsymbol{u}$ external to C (averaged about the circumference of the core) and the velocity $\boldsymbol{v}$ of material particles on C, or

$$\boldsymbol{q} = (\boldsymbol{u} - \boldsymbol{v}) - [(\boldsymbol{u} - \boldsymbol{v})\cdot\lambda_3]\lambda_3 \tag{9}$$

The first term on the right-hand side of Eq. (8) represents the Kutta-Joukowski lift force, and the last two terms represent the "added mass" and "buoyancy" forces due to external flow relative to the vortex core (an explicit form for these latter two forces is given in Ref. 15). The scalar ℓ is related to the average pressure p_s on the core lateral surface and is given by

$$\ell = -p_s/\rho^*\sigma \simeq \frac{\Gamma^2}{8\pi^2\sigma^3} - \frac{p_\infty}{\rho^*\sigma} \tag{10}$$

where p_∞ is the pressure in the normal plane to C far away from the vortex core. The second term on the right-hand side of Eq. (8) is related to the axial force that arises from a gradient in core area along C, as well as a force in the principal normal direction that is canceled by an internal force associated with $\boldsymbol{n}$.

The contact force $\boldsymbol{n}$ in Eq. (6) is necessary to maintain the constraint that the fluid from which the vortex is composed is incompressible. An expression for $\boldsymbol{n}$ is derived in Ref. 15 as

$$\boldsymbol{n} = -p\sigma^2 \underset{\sim}{\lambda}_3 \tag{11}$$

where $p(\xi, t)$ is a Lagrange multiplier. Using an additional equation related to the rate of change of the radial momentum within the core, an expression for p is derived in Ref. 15 as

$$p = -\pi\rho^*\sigma\ell + \frac{\pi}{4}\rho^*\sigma\left(\ddot{\sigma} - \frac{\Gamma^2}{4\pi^2\sigma^3}\right) \tag{12}$$

The external velocity $\boldsymbol{u}$ (averaged about the core circumference), which is needed to determine $\boldsymbol{f}$, is composed of the sum of a part $\boldsymbol{u}_I$ due to the self-induced velocity of the curved vortex and an additional prescribed part $\boldsymbol{u}_E$. The Biot-Savart equation [Eq. (1)] is used to determine $\boldsymbol{u}_I$, where the cutoff constant δ_R is given by the expression usually recorded in the literature for a hollow core vortex since the internal forces within the core are accounted for elsewhere in the theory. The part $\boldsymbol{u}_E$ is simply the value of any external potential flowfield evaluated along the curve C.

The vortex velocity $\boldsymbol{v}$ and the external velocity fields $\boldsymbol{u}_I$ and $\boldsymbol{u}_E$ can be written in terms of their components along the principal normal, binormal, and tangent directions to C as

$$\boldsymbol{v} = u\underset{\sim}{\lambda}_1 + v\underset{\sim}{\lambda}_2 + w\underset{\sim}{\lambda}_3, \qquad \boldsymbol{u}_I = u_r\underset{\sim}{\lambda}_I + v_I\underset{\sim}{\lambda}_2 + w_I\underset{\sim}{\lambda}_3$$
$$\boldsymbol{u}_E = u_E\underset{\sim}{\lambda}_1 + v_E\underset{\sim}{\lambda}_2 + w_E\underset{\sim}{\lambda}_3 \tag{13}$$

The vortex momentum equation [Eq. (6)] can be written in terms of its components, using Eqs. (2), (7), and Eqs. (10–13) as

$$\rho(\dot{u} + v\dot{\underset{\sim}{\lambda}}_2\cdot\underset{\sim}{\lambda}_1 + w\dot{\underset{\sim}{\lambda}}_3\cdot\underset{\sim}{\lambda}_1) = \rho^*\Gamma(v_I + v_E - v)$$
$$+ X_1 + Y_1 - \frac{\rho\kappa\sigma}{4}\left(\ddot{\sigma} - \frac{\Gamma^2}{4\pi^2\sigma^3}\right) \tag{14}$$

$$\rho(\dot{v} + u\dot{\underset{\sim}{\lambda}}_1\cdot\underset{\sim}{\lambda}_2 + w\dot{\underset{\sim}{\lambda}}_3\cdot\underset{\sim}{\lambda}_2) = -\rho^*\Gamma(u_I + u_E - u) + X_2 + Y_2 \tag{15}$$

$$\rho(\dot{w} + u\dot{\underset{\sim}{\lambda}}_1\cdot\underset{\sim}{\lambda}_3 + v\dot{\underset{\sim}{\lambda}}_2\cdot\underset{\sim}{\lambda}_3)$$
$$= X_3 + Y_3 - 2\rho\ell\frac{\partial\sigma}{\partial s} + \frac{\partial}{\partial s}\left(\rho\sigma\ell - \frac{\sigma^3\ddot{\sigma}}{4}\right) \tag{16}$$

which provide differential equations for u, v, and w. The core radius σ is obtained by solution of Eq. (5), and equations for curvature κ and torsion τ are obtained from the kinematical requirement that $\dot{\underset{\sim}{\lambda}}_1\cdot\underset{\sim}{\lambda}_1$, $\dot{\underset{\sim}{\lambda}}_2\cdot\underset{\sim}{\lambda}_2$, and $\dot{\underset{\sim}{\lambda}}_3\cdot\underset{\sim}{\lambda}_3$ vanish (the latter of which is satisfied identically under the assumption of incompressibility of the core fluid). Expressions for $\dot{\underset{\sim}{\lambda}}_i\cdot\underset{\sim}{\lambda}_j$ in terms of the intrinsic properties of the vortex are given in Ref. 15.

Although the previous analysis yields a well-posed set of equations for the vortex motion, the equations are unfortunately rather stiff for numerical solution. This difficulty arises from the observation that the highest-order time derivative terms in u and v on the left-hand sides of Eqs. (14) and (15) are smaller by a factor of at least $\varepsilon = \max(\sigma/L)$ than the largest terms on the right-hand sides of these equations.

Let us assume (based on solutions for axial and helical wave propagation) that the lateral velocity components u and v are of $\mathcal{O}(\Gamma/L)$, the axial velocity w is of $\mathcal{O}(w_0) \leq \mathcal{O}(\Gamma/\sigma_0)$, and time t is of $\mathcal{O}(\sigma_0 L/\Gamma)$, where L and σ_0 are the typical length scales of ξ and σ, respectively. Retaining only leading-order terms in ε and using Eq. (10), Eqs. (14–16) reduce to

$$\pi\sigma^2\kappa w^2 = \Gamma(v_I + v_E - v) + \frac{\kappa\Gamma^2}{16\pi} \tag{17}$$

$$0 = -\Gamma(u_I + u_E - u) \tag{18}$$

$$\pi\sigma^2\dot{w} = -\frac{\Gamma^2}{4\pi\sigma}\frac{\partial\sigma}{\partial s} - \frac{\pi\sigma^2}{\rho^*}\frac{\partial p_\infty}{\partial s} \tag{19}$$

Defining the relative axial velocity component w_A by $w_A = w - w_I - w_E$, and noting that the order of magnitude of w_I and w_E is smaller than that of w_A by a factor of ε, solutions for the vortex velocity components u, v and w can be obtained from Eqs. (17–19), valid to leading order in ε, as

$$u = u_I + u_E \tag{20}$$

$$v = v_I + v_E + \frac{\kappa\Gamma}{16\pi} - \frac{\pi\sigma^2\kappa w_A^2}{\Gamma} \tag{21}$$

$$w = w_A + w_I + w_E \tag{22}$$

where w_A is obtained from solution of the differential equation

$$\pi\sigma^2\dot{w}_A = -\frac{\Gamma^2}{4\pi\sigma}\frac{\partial\sigma}{\partial s} - \frac{\pi\sigma^2}{\rho^*}\frac{\partial p_\infty}{\partial s} \tag{23}$$

The external pressure gradient term in Eq. (23) is usually negligible. For instance, the self-induced pressure gradient of a curved vortex is smaller by a factor of $\mathcal{O}(\varepsilon^2)$ than the other terms in Eq. (23). However, for certain flows in which a sufficiently strong pressure gradient is maintained over a long portion of the vortex axis, the amplitude and stability of axial and helical waves on the vortex can be significantly affected, as discussed in Ref. 17. A special case of Eqs. (20–22) was derived previously by Moore and Saffman[13] under the assumption that w_A and σ are uniform along C and that u_I and w_I vanish.

We note that the solutions for u and v in Eqs. (20) and (21) are algebraic once the self-induced velocity $\boldsymbol{u}_I$ is obtained from the integral Eq. (1). The long-wave theory thus requires that u and v be initially chosen to satisfy Eqs. (20) and (21), whereas the more general equations [Eq. (14) and (15)] admit arbitrary initial conditions for u and v. This difference is a consequence of the neglect in the long-wave theory of the highest-order time derivatives in Eqs. (14) and (15).

When u, v, and $\partial p_\infty/\partial s$ vanish, the pair of equations [Eqs. (5) and (23)] for σ and w_A form a conservative hyperbolic system that is of a form equivalent to the one-dimensional gasdynamics equations. This system admits an implicit solution by the method of characteristics,[14] and there are numerous well-tested algorithms for numerical solution of the gasdynamics equations[18] that include calculation of shocks. For the problem of blade-vortex interaction, the effect of the blade on the vortex motion is accounted for through the term $\boldsymbol{u}_E$, which is coupled with the solution for the flow past the vortex. A demonstration of the use of the long-wave vortex filament theory for numerical calculation of normal blade-vortex interaction is given in Part 2.

III. Solution for the Cutting of a Straight Vortex by a Flat Plate

We now consider the problem of impulsive cutting of a straight vortex by a flat plate (at angle of attack α), for which an explicit solution of Eqs. (5) and (23) for σ and w_A can be obtained using the method of characteristics. This system admits Riemann invariants of the form

$$J^+ = w - 2c = \text{const on } \frac{ds}{dt} = w + c \ (C^+ \text{ characteristics})$$
$$J^- = w + 2c = \text{const on } \frac{ds}{dt} = w - c \ (C^- \text{ characteristics}) \tag{24}$$

and the "vortex sound" speed c is given by

$$c = (\Gamma^2/8\pi^2\sigma^2)^{1/2} \tag{25}$$

The vortex is assumed to have an initial axial velocity w_0 and core radius σ_0 before cutting by the blade, and cutting is assumed to occur instantaneously at $t = 0$. The ambient value c_0 of c is the value of c at $\sigma = \sigma_0$. A schematic of the flow geometry is shown in Fig. 1, in which a Cartesian coordinate system is used with origin at the leading edge of the blade. The vortex axis is parallel to the z direction, such that the variable s in Eq. (2) is identical to the z coordinate, and the blade is traveling at a constant speed U in the negative x direction with a (positive) constant angle of attack α. The solution given in the present paper, which concerns the behavior of the vortex following cutting by the blade, will be obtained separately in the upper and lower halves of the vortex.

At time $t > 0$, the vortex intersects the blade at a point $z_0 = -\alpha Ut$ along the z axis, such that this intersection point travels along the vortex axis with speed $-\alpha U$. The axial velocity component within the vortex at this intersection point might be different from the intersection point velocity due to suction of boundary-layer fluid from the blade into the vortex core. This suction velocity, with magnitude w_s directed away from the blade, can be estimated using viscous flow solutions for the case of a stationary Rankine vortex over a stationary flat plate,[19,20] which suggest (for laminar flow) an expression of the form

$$w_s = (\Gamma\nu/\pi\sigma^2)^{1/2} \tag{26}$$

Both turbulence and movement of the blade relative to the vortex would be expected to influence w_s, the latter being particularly important for values of $2\pi\sigma U/\Gamma$ of $\mathcal{O}(1)$ or larger. For typical applications with helicopter rotors, for instance, data such as that of Tung et al.[21] indicate values for this ratio of about 0.3. In the present paper, we merely assume that w_s is some prescribed function of σ and Γ.

Since the core radius and vortex axial velocity are expected to be different on opposite sides of the blade surface, we use a superscript plus sign to denote the upper side of the blade surface and a superscript minus sign to denote the lower side of the blade surface. The boundary conditions on the axial flow at $z = z_0$, denoted by w_B^+ and w_B^-, are given by

$$\begin{aligned} w_B^+ &\equiv w(z_0^+) = w_s(\sigma^+) - \alpha U \\ w_B^- &\equiv w(z_0^-) = -w_s(\sigma^-) - \alpha U \end{aligned} \tag{27}$$

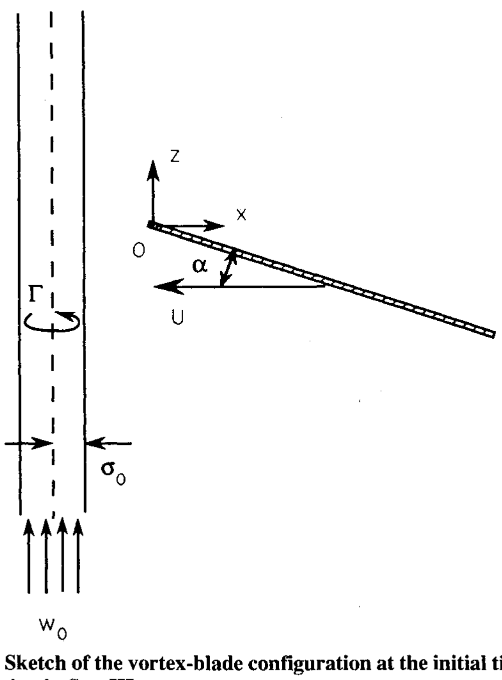

Fig. 1 Sketch of the vortex-blade configuration at the initial time t for the solution in Sec. III.

The solution of this problem is mathematically similar to that of impulsive piston motion in a shock tube. The form of the solution depends on whether the flow on each side of the blade is a compression or an expansion of the vortex axis C. On the upper half of the vortex, the flow will be a compression if $w_0 < w_B$ and an expansion otherwise. In the lower half of the vortex, the flow will be an expansion if $w_0 < w_B$ and a compression otherwise. (We require that $\alpha \geq 0$ to distinguish between the upper and lower halves of the vortex, but w_0 may have any value.) In the case that the flow is an expansion, an "expansion wave" will propagate away from the blade such that the rear of the wave and the front of the wave are located at z_R and z_F, respectively, where for the upper half of the vortex

$$z_R(t) = [c_0 + 1/2(w_0 + w_B)]t, \qquad z_F(t) = (c_0 + w_0)t \tag{28}$$

and for the lower half of the vortex

$$z_R(t) = -[c_0 - 1/2(w_0 + w_B)]t, \qquad z_F(t) = -(c_0 - w_0)t \tag{29}$$

For $|z| > |z_F|$, w and σ have the same values w_0 and σ_0 as for the undisturbed vortex. For $|z| < |z_R|$, w is equal to its value w_B at the blade surface, and the wave speed c_B is given by

$$c_B = c_0 \pm 1/2(w_0 - w_B) \tag{30}$$

where the plus sign is for the upper half vortex and the minus sign is for the lower half vortex. A solution for the core radius σ_B near the blade on the expansion side can be obtained from Eqs. (25) and (30) as

$$\frac{\sigma_B}{\sigma_0} = \frac{c_0}{c_B} = \left(1 \pm \frac{1}{2}\frac{w_0 - w_B}{c_0}\right)^{-1} \tag{31}$$

where the plus and minus signs have the same association as in Eq. (30). Within the expansion wave ($|z_R| < |z| < |z_F|$), solutions for w and c are given for any z and t by

$$w = -w_0 - 2c_0 + 2z/t, \qquad c = 2c_0 + w_0 - z/t \tag{32}$$

for the upper half vortex and

$$w = -w_0 + 2c_0 + 2z/t, \qquad c = 2c_0 - w_0 + z/t \tag{33}$$

for the lower half vortex.

In an expansion, the difference $|z_F| - |z_R|$ increases, and the expansion wave spreads out with time, but for a compression this difference decreases with time, and the axial waves build up into a vortex shock. In the current solution, we assume instantaneous cutting of the vortex so that the shock forms at the blade at $t = 0$ and propagates away from the blade with speed W for $t > 0$ with constant strength. If we were instead to allow the cutting time to have some finite value approximately equal to $2\sigma_0/U$, the shock strength would increase within a distance of about $2W\sigma_0/U$ from the blade, after which the strength would be nearly constant.

For the case of instantaneous cutting, the values of σ and w between the blade and the shock are denoted by σ_B and w_B, where w_B is given by Eq. (27), whereas ahead of the shock σ and w are simply given by the ambient values σ_0 and w_0. Jump conditions in mass and momentum across a vortex shock[14,15] are given by

$$\begin{aligned} [\![\sigma^2(w - W)]\!] &= 0 \\ [\![\sigma^2 w(w - W)]\!] &= -\frac{\Gamma^2}{4\pi^2} [\![\ln(\sigma/\sigma_0)]\!] \end{aligned} \tag{34}$$

where the notation $[\![f]\!]$ denotes the value of some quantity f slightly above the shock minus the value of f slightly below the shock. The shock speed W can be determined from Eq. (34), and the condition that the jump in w across the shock is given by $[\![w]\!] = \pm(w_0 - w_B)$, where again the plus sign is for the upper half vortex and the negative sign for the lower half vortex. The remaining jump condition in Eq. (34) can be used to provide an equation for the core radius σ_B near the blade. By this approach, we obtain an explicit solution for W and a nonlinear equation that can be solved iteratively for σ_B as follows:

$$W = w_B \pm \frac{\sigma_0}{2\pi\sigma_B}\left[\frac{\Gamma^2 \ln(\sigma_B/\sigma_0)}{\sigma_B^2-\sigma_0^2}\right]^{1/2} \tag{35}$$

$$4\pi^2\sigma_0^2\sigma_B^2(w_B-w_0)^2 = \Gamma^2(\sigma_B^2-\sigma_0^2)\ln(\sigma_B/\sigma_0) \tag{36}$$

where the plus sign in Eq. (35) is for the upper half vortex, and the minus sign is for the lower half vortex. We note that the solution given here for a vortex shock satisfies the full axial momentum equation [Eq. (16)] for the case of a straight vortex and that the long-wave approximation is consistent with the solution obtained for expansion waves.

A plot of (σ_B/σ_0) vs $(w_B-w_0)^2/c_0^2$ is given in Fig. 2 for both the expansion and compression waves, as predicted from Eq. (31) and the iterative solution of Eq. (36). With an increase in $(w_B - w_0)^2/c_0^2$, the core radius near the blade decreases slightly from its ambient value in an expansion, whereas in a compression a very steep increase in core radius near the blade is observed. In fact, for large values of (σ_B/σ_0), Eq. (36) indicates that the core radius near the blade increases exponentially in a compression as

$$\frac{\sigma_B}{\sigma_0} \sim \exp\left[\frac{2\,(w_B-w_0)^2}{c_0^2}\right] \tag{37}$$

IV. Vortex Force on the Blade

The magnitude of the force on the blade due to the vortex after it is cut, which acts in the direction of the tangent to the vortex axis, is denoted by F_B. The vortex force is caused both by pressure differences across the blade surface (which arise from differences in core radius) and by momentum of the fluid sucked into the vortex from the blade boundary layer. It is a simple matter to show that the integrated pressure over the core of a Rankine vortex is independent of the core radius (provided that the core radius is independent of time). Since the upper and lower halves of the vortex have the same circulation, the net pressure force on the blade must come from the additional force exerted on the blade on the side with the smallest core radius by the fluid external to the vortex core, in a ring of width equal to the difference between the core radius on the two sides of the blade. Integrating the pressure for a potential vortex over this ring gives the net pressure force magnitude F_p as

$$F_p = \frac{\rho^*\Gamma^2}{4\pi}\ln\left(\frac{\sigma^+}{\sigma^-}\right) \tag{38}$$

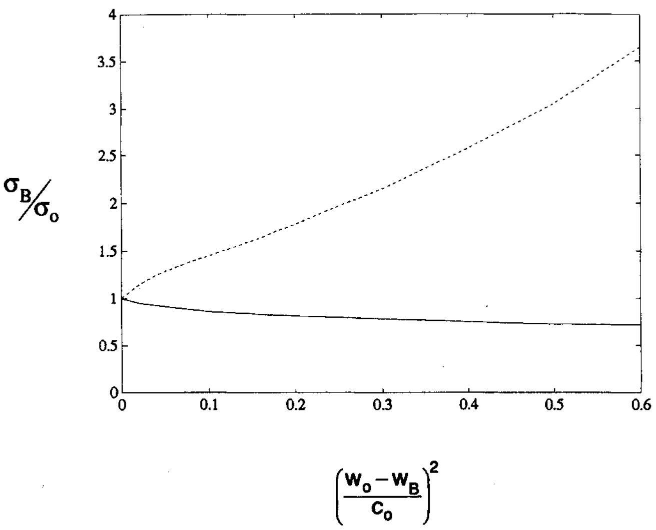

Fig. 2 Plot of the ratio of core radius σ_B near the blade to the ambient core radius σ_0 both for expansion cases (solid curve) and for compression cases (dashed curve) as predicted from Eqs. (31) and (36).

where σ^+ and σ^- denote the values of the core radius on the upper and lower sides of the blade, respectively. The force magnitude F_s on the blade due to suction of boundary-layer fluid into the vortex is given by

$$F_s = -[\![\rho^*\pi\sigma^2 w_s^2]\!] \tag{39}$$

where $[\![f]\!] = f^+ - f^-$ is the jump across the blade of any quantity f and w_s is the (positive) suction speed into the vortex. The total blade force is given by the sum of these two effects, or

$$F_B = F_p + F_s = \left[\!\!\left[\frac{\rho^*\Gamma^2}{4\pi}\ln(\sigma) - \rho^*\pi\sigma^2 w_s^2\right]\!\!\right] \tag{40}$$

This same result, Eq. (40), can be obtained alternatively using the general jump conditions in mass and axial momentum from Ref. 15, given by

$$M = [\![\rho(w-W)]\!], \qquad F = [\![\rho w(w-W) - n]\!] \tag{41}$$

In Eq. (41), the fluid axial velocity within the vortex core at the point of intersection with the blade is denoted by w. The variables M and F on the left-hand sides of Eq. (41) are the rate of mass supply and the axial component of the rate of momentum supply, respectively, at the discontinuity. The mass supply rate M arises from the suction of blade boundary-layer fluid into the vortex, and the axial momentum supply rate F arises (partially) from the force on the vortex by the blade. From Eqs. (10–12), the jump in $n = \boldsymbol{n}\cdot\boldsymbol{\lambda}_3$ over the discontinuity vanishes when the core radius is constant with time and has vanishing gradient on both sides of the discontinuity, as is the case for the solution given in the previous section. We now consider a steady translation of the observational frame with a constant axial velocity b, such that in the new frame (denoted with a superscript plus sign)

$$\rho^+ = \rho, \quad M^+ = M, \quad w^+ = w + b, \quad W^+ = W + b, \quad n^+ = n \tag{42}$$

Since the jump conditions of Eq. (41) must be invariant under steady translation of the observational frame, it follows that

$$F^+ = F + Mc \tag{43}$$

To satisfy (43), we decompose F as

$$F = F_I + MW \tag{44}$$

where F_I is invariant under translation and can be identified as the negative of the force on the blade due to the fluid within the vortex core. Using Eqs. (41) and (44), and setting the jump in n equal to zero, we find that

$$F_I = [\![\rho(w-W)^2]\!] \tag{45}$$

Since $w - W = \pm w_s$, F_I in Eq. (45) reduces to the negative of the "suction force" F_s given previously in Eq. (39). The force on the blade due to the fluid external to the vortex core, denoted previously by F_p, can be obtained by integrating the external body force ρf per unit length acting on the core surface [given by Eq. (8)] over the discontinuity, which gives the same result as that recorded in Eq. (38). The total force on the blade is the sum of the internal and external parts (i.e., $F_B = F_p - F_I$) as indicated in Eq. (40).

V. Conclusions

A simplified long-wave theory is presented in the paper for motion of vortex filaments with variable core area. In this theory, the components of filament motion normal to the vortex are determined by the Biot-Savart integral (with cutoff for a hollow core vortex) plus two additional terms in the direction of the binormal to the vortex axis, which are related to the integrated pressure across the core and the centrifugal force due to axial flow within the core. The velocity of material particles along the vortex axis is determined by an equation balancing the rate of change of axial momentum with the restoring force due to the pressure gradient on the core lateral surface (caused by an axial gradient in core crosssectional area). This equation has a form analogous to that for one-dimensional gasdynamics, and admits "vortex shocks" on the core.

The long-wave theory is used to obtain a simple solution for the problem of instantaneous cutting of a straight vortex filament by a flat plate, which involves propagation of a vortex shock and a vortex expansion wave away from the blade on opposite sides of the vortex. This solution is used to derive an expression for the force F_B that the vortex exerts on the blade, which depends on both the rate of suction of blade boundary-layer fluid into the vortex and the ratio of vortex core radius on opposing sides of the blade.

The solution indicates that as the blade cuts the vortex, over a time interval of approximate duration $2\sigma_0/U$, the magnitude of the force on the blade changes by an amount F_B, an expression for which can be obtained by substituting into Eq. (40) the results for the core radius near the blade from Eqs. (31) and (36), for the expansion wave and shock, respectively. A rough estimate of the maximum rate of change of the force on the blade would then be $UF_B/2\sigma$, which could be used to obtain an expression for the noise generated by the blade-vortex interaction (see Ref. 22). Before recommending such a course, however, we note that there are a number of issues that remain to be resolved. First, although we have obtained a solution for the vortex force on the blade just before and just after cutting, we have not shown that the force varies monotonically between these values (or, rather, that the force on the blade doesn't have a "spike" just at the instant of cutting). Second, our solution doesn't account for bending of the vortex due to interaction with the blade, which might be expected to occur for cases where the blade is much thicker than the vortex core. The structure of the vortex shock is also not yet understood; in particular, whether it is an axisymmetric disturbance or whether some instability causes bending waves on the vortex downstream of the shock. Some of these issues will be taken up in subsequent papers.

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